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2009 J. Phys. A: Math. Theor. 42 475305

(http://iopscience.iop.org/1751-8121/42/47/475305)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 475305 (12pp) doi:10.1088/1751-8113/42/47/475305

Quantum state transfer in a q-deformed chain

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Received 13 February 2009, in final form 5 October 2009
Published 6 November 2009
Online at stacks.iop.org/JPhysA/42/475305

Abstract
We investigate the quantum state transfer in a chain of particles satisfying the q-deformed oscillators algebra. This general algebraic setting includes the spin chain and the bosonic chain as limiting cases. We study conditions for perfect state transfer depending on the number of sites and excitations on the chain. They are formulated by means of irreducible representations of a quantum algebra realized through Jordan–Schwinger maps. Playing with deformation parameters, we can study the effects of nonlinear perturbations or interpolate between the spin and bosonic chains.

PACS numbers: 03.67.Hk , 03.65.Fd , 02.20.Uw

1. Introduction

Spatially distributed interacting quantum systems can provide a means of transferring quantum information from one place to another. This possibility relies on quantum interference effects arising from the evolution of the whole system. An example along this line is given by a chain of spin-$\frac{1}{2}$ systems where perfect state transfer from one to another end can be realized [1, 2]. Another example is given by a chain of harmonic oscillators [3, 4]. These two examples come, under the mathematical point of view, from the realizations of two different algebras (the Lie algebra su(2) and the Heisenberg–Weyl algebra) corresponding to fermionic and bosonic commutation relations. The latter can be seen as two limit cases of more general commutation relations involving deformed algebras parameterized by one continuous parameter [5, 6]. Because of the increasing interest on the topic of state transfer in a chain of quantum systems (see e.g. [7]), it would be interesting to investigate the state transfer in a more general algebraic setting. In perspective, that could pave the way for a systematic study of the role of algebraic structures in the problem of state transfer. Moreover, the deformed algebraic setting can be used as a formal way of describing nonlinear interaction in the quantum chain.

We start by considering a chain of $n+1$ sites described by a nearest-neighbor Hamiltonian of the kind

$$H = \sum_{j=1}^{n} J_j \frac{{a_j}^{\dagger} a_{j+1} + {a_{j+1}}^{\dagger} a_j}{2}, \tag{1}$$

where J_j are the coupling constants. ${a_j}^{\dagger}$, a_j are ladder operators whose algebraic properties determine the nature of the quantum chain. Their canonical commutation and anticommutation relations respectively define a bosonic and a fermionic quantum chain. Moreover, the fermionic chain can be mapped, via the Jordan–Wigner map, onto a chain of spin-1/2 [8].

Here we consider a quantum chain of q-deformed oscillators. Several kinds of deformed oscillator algebras have been introduced and studied in the literature. Here we are mainly concerned with the complex (associative unital) algebra, called the (*symmetric*) *q-oscillator algebra* and denoted by $\mathcal{A}_q$ [9]. Each site of the quantum chain is endowed with a copy of $\mathcal{A}_q$, with four generators $a^{\dagger}$, a, q^N, q^{-N} subject to the relations

$$aa^{\dagger} - qa^{\dagger}a = q^{-N}. \tag{2}$$

$$q^{-N}q^N = q^N q^{-N} = 1, \qquad q^N a^{\dagger} = qa^{\dagger}q^N, \qquad q^N a = q^{-1}aq^N. \tag{3}$$

From (2), (3) the following properties can easily be derived:

$$a^{\dagger}a = [N], \qquad aa^{\dagger} = [N+1], \tag{4}$$

where the notation $[N]$ indicates the q-number N, defined as

$$[N] := \frac{q^N - q^{-N}}{q - q^{-1}}. \tag{5}$$

It is suitable to recall that the algebra $\mathcal{A}_q$ is a $*$-algebra with involution such that $a^* = a^{\dagger}$ and $(q^N)^* = q^N$. A key role is played by the representation T of $\mathcal{A}_q$ on a Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|m\rangle : m \in \mathbb{N}\}$, defined as

$$T(a)|m\rangle = \sqrt{[m]}|m-1\rangle, \qquad T(a^{\dagger})|m\rangle = \sqrt{[m+1]}|m+1\rangle, \qquad T(N)|m\rangle = m|m\rangle. \tag{6}$$

If $\mathcal{D}$ denotes the dense linear subspace of $\mathcal{H}$ spanned by the vectors $|m\rangle$, then the representation T becomes the Fock representation of the q-oscillator algebra A_q, that is, the $*$-representation of the $*$-algebra $\mathcal{A}_q$ on $\mathcal{D}$.

For our investigation, we need to introduce the algebra $\mathcal{A}_q^{\text{ext}}$ obtained by adjoining formally elements $q^{N/2}$ and $q^{-N/2}$ to $\mathcal{A}_q$. Then, a chain of two sites can be represented by the tensor product $\mathcal{A}_q^{\text{ext}\otimes 2}$ of two q-oscillator algebras $\mathcal{A}_q^{\text{ext}}$ whose generators are denoted by $a_1\, a_1^{\dagger}$, $q^{N_1/2}$, $q^{-N_1/2}$, $a_2\, a_2^{\dagger}$, $q^{N_2/2}$, $q^{-N_2/2}$. It is relevant to note that every element of the set $a_1\, a_1^{\dagger}$, $q^{\pm N_1/2}$ commutes with any element from $a_2\, a_2^{\dagger}$, $q^{\pm N_2/2}$. The great difference with the classical case is that the q-oscillator algebra (generated by the deformed relations) does not realize any matrix algebra but realizes, by the deformed Jordan–Schwinger map, a suitable quantum algebra which constitutes our mathematical framework. As a consequence, we will see that the relations for perfect state transfer can be formulated by its irreducible representations.

This paper is organized as follows. In section 2, we present the quantum algebra $U_q(\mathrm{sl}_{n+1})$ for $n \geqslant 1$ by discussing some crucial properties and emphasizing its (deformed) Jordan–Schwinger realization in terms of q-oscillator algebras. In section 3, the irreducible representations of $U_q(\mathrm{sl}_{n+1})$ are presented by composing the Jordan–Schwinger map with the Fock representation of $\mathcal{A}_q$. This framework allows us to represent the physical system of the chain with $n + 1$ sites. Section 4 is devoted to the study of state transfer through a chain of q-deformed oscillators. For the case of a chain of spin-1/2, fermions, or bosons, the Hamiltonian function with nearest-neighbor interaction as (1) allows perfect state transfer if the coupling constants J_j are suitably chosen. We consider the efficacy, for the issue of quantum state transfer, of one of the choices in the case of a chain of q-deformed oscillators. Conclusions and possible physical applications are drawn in section 5.

2. The quantum algebra $U_q(\mathrm{sl}_{n+1})$

Before analyzing the issue of state transfer through a chain of q-deformed oscillators, we fix our mathematical setting.

Let q be a complex number such that $q \neq 0$ and $q^2 \neq 1$. We first consider the quantized universal enveloping algebra $U_q(\mathrm{sl}_2)$ of the Lie algebra sl_2 of all traceless 2×2 matrices with coefficients in the field of complex numbers $\mathbb{C}$. $U_q(\mathrm{sl}_2)$ can be described as the associative algebra with the unity over $\mathbb{C}$ with four generators E, F, K, K^{-1} satisfying the defining relations

$$KK^{-1} = K^{-1}K = 1, \qquad KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \tag{7}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{8}$$

It can be shown by induction that the relations (7) and (8) imply for every positive integer s and t the formulae

$$[E, F^t] = [t]F^{t-1}\frac{Kq^{1-t} - K^{-1}q^{t-1}}{q - q^{-1}}, \tag{9}$$

$$[E^s, F] = [s]E^{s-1}\frac{Kq^{s-1} - K^{-1}q^{1-s}}{q - q^{-1}}. \tag{10}$$

A key property of the algebra $U_q(\mathrm{sl}_2)$ is that it carries a Hopf algebra structure. Indeed, we can remind that there exists a unique Hopf algebra structure on $U_q(\mathrm{sl}_2)$ with comultiplication Δ, counit ε, antipode S

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K, \tag{11}$$

$$S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF, \quad \varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0. \tag{12}$$

From now on, we refer to this algebra endowed with the Hopf algebra structure as the quantum algebra $U_q(\mathrm{sl}_2)$.

The quantum algebra $U_q(\mathrm{sl}_2)$ could be supposed to be a quantum analog of the enveloping algebra $U(\mathrm{sl}_2)$ of the Lie algebra sl_2. In fact, $U_q(\mathrm{sl}_2)$ shares two main properties with the classical one: it has no zero divisors (see e.g. [10, proposition 1.8]) and it has a Poincaré–Birkhoff–Witt-type basis (see e.g. [11, section 3.1]), that is, $U_q(\mathrm{sl}_2)$ as $\mathbb{C}$-vector space is generated by the basis $\{E^sK^lF^t \mid s, t \in \mathbb{N} - \{0\}, l \in \mathbb{Z}\}$.

Unfortunately we cannot straightforwardly recover $U(\mathrm{sl}_2)$ from $U_q(\mathrm{sl}_2)$ by setting $q = 1$ (as it happens at the level of representation theory) but by considering the limit of $q \to 1$ of a slight reformulation of $U_q(\mathrm{sl}_2)$ at least for q not a root of unity (see e.g. [11, section 3.1.3]). For our goals, it is relevant to equip $U_q(\mathrm{sl}_2)$ with an involution $* : U_q(\mathrm{sl}_2) \to U_q(\mathrm{sl}_2)$ which turns

$U_q(\mathrm{sl}_2)$ into a Hopf $*$-algebra, usually called the real form of $U_q(\mathrm{sl}_2)$ and denoted (slightly abusing the notation) again by $U_q(\mathrm{sl}_2)$.

The realization of $U_q(\mathrm{sl}_2)$ in terms of the q-oscillator algebra $\mathcal{A}_q^{\mathrm{ext}\otimes 2}$ (with generators $a_1 a_1{}^\dagger$, $q^{\pm N_1/2}$, $a_2 a_2{}^\dagger$, $q^{\pm N_2/2}$) can be allowed by the (deformed) Jordan–Schwinger map $\mathrm{JS}_q : U_q(\mathrm{sl}_2) \to \mathcal{A}_q^{\mathrm{ext}\otimes 2}$ defined (similarly to the classical case) as

$$\mathrm{JS}_q(E) = a_1^\dagger a_2, \qquad \mathrm{JS}_q(F) = a_2^\dagger a_1, \qquad \mathrm{JS}_q(K) = q^{(N_1 - N_2)/2}. \tag{13}$$

By composing the (unique) algebra homomorphism JS_q with the Fock representation of $\mathcal{A}_q^{\mathrm{ext}\otimes 2}$, irreducible representations of $U_q(\mathrm{sl}_2)$ can be obtained. These representations give the right setting where the relations for the state transfer in a chain with two sites can be formulated. The same thing can be repeated when we consider a chain with $n+1$ sites. Hence, we are going on introducing the related quantum algebra, that is, the universal enveloping algebra $U_q(\mathrm{sl}_{n+1})$ of the Lie algebra sl_{n+1} of all traceless $n \times n$ matrices.

First, consider the Lie algebra sl_{n+1} for $n \geqslant 1$ and the root system Φ of sl_2 with a basis Π formed by n roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. According to the scalar product $(\cdot, \cdot)$ on the vector space generated by Φ, we have that $(\alpha, \alpha) = 2$ for every (short) root α of Φ.

The quantized enveloping algebra of sl_{n+1} is a $\mathbb{C}$-algebra $U_q(\mathrm{sl}_{n+1})$ with $4n$ generators E_{α_j}, F_{α_j}, K_{α_j}, $K_{\alpha_j}^{-1}$ with $j = 1, \ldots, n$ and relations

$$\begin{array}{lr}
K_{\alpha_j} E_{\alpha_l} K_{\alpha_i}^{-1} = q^2 E_{\alpha_l} \quad \text{and} \quad K_{\alpha_j} F_{\alpha_l} K_{\alpha_j}^{-1} = q^{-2} F_{\alpha_l} & (j = l) \\
K_{\alpha_j} E_{\alpha_l} K_{\alpha_j}^{-1} = q^{-1} E_{\alpha_l} \quad \text{and} \quad K_{\alpha_j} F_{\alpha_l} K_{\alpha_j}^{-1} = q F_{\alpha_l} & (|j-l| = 1) \\
K_{\alpha_j} E_{\alpha_l} K_{\alpha_j}^{-1} = E_{\alpha_l} \quad \text{and} \quad K_{\alpha_j} F_{\alpha_l} K_{\alpha_j}^{-1} = F_{\alpha_l} & (|j-l| \geqslant 2) \\
K_{\alpha_j} K_{\alpha_l} = K_{\alpha_l} K_{\alpha_j} \quad \text{and} \quad E_{\alpha_j} F_{\alpha_l} - F_{\alpha_l} E_{\alpha_j} = \delta_{jl} \frac{K - K^{-1}}{q - q^{-1}} & \\
E_{\alpha_j} E_{\alpha_l} = E_{\alpha_l} E_{\alpha_j} \quad \text{and} \quad F_{\alpha_j} F_{\alpha_l} = F_{\alpha_l} F_{\alpha_j} & (|j-l| \geqslant 2) \\
E_{\alpha_j}^2 E_{\alpha_l} - (q + q^{-1}) E_{\alpha_j} E_{\alpha_l} E_{\alpha_j} + E_{\alpha_l} E_{\alpha_j}^2 = 0 & (|j-l| = 1) \\
F_{\alpha_j}^2 F_{\alpha_l} - (q + q^{-1}) F_{\alpha_j} F_{\alpha_l} F_{\alpha_j} + F_{\alpha_l} F_{\alpha_j}^2 = 0 & (|j-l| = 1).
\end{array}$$

When $n = 1$, we obviously obtain the relations (7), (8) of the quantized universal enveloping algebra of sl_2. Similar to the case of $U_q(\mathrm{sl}_2)$, a Hopf algebra structure is carried by $U_q(\mathrm{sl}_{n+1})$ which is therefore treated as a quantum algebra: to define the comultiplication, the antipode and the counit it is enough to apply the same relations (12), (11) (described for $U_q(\mathrm{sl}_2)$) to the generators E_{α_j}, F_{α_j}, K_{α_j}, $K_{\alpha_j}^{-1}$, with $j = 1, \ldots, n$. Furthermore, we can endow $U_q(\mathrm{sl}_{n+1})$ with an involution $* : U_q(\mathrm{sl}_{n+1}) \to U_q(\mathrm{sl}_{n+1})$ which turns $U_q(\mathrm{sl}_{n+1})$ into a Hopf $*$-algebra.

It is worth noting that when $n > 1$, it is always possible to consider a subalgebra of $U_q(\mathrm{sl}_{n+1})$ which is isomorphic to $U_q(\mathrm{sl}_2)$. More precisely, $\forall i$ the tuple of generators $(E_{\alpha_j}, F_{\alpha_j}, K_{\alpha_j}, K_{\alpha_j}^{-1})$ satisfies the same relations (7), (8) of $U_q(\mathrm{sl}_2)$, so we have for each $\alpha_j \in \Pi$ the homomorphism $U_q(\mathrm{sl}_2) \to U_q(\mathrm{sl}_{n+1})$ that takes E to E_{α_j}, F to F_{α_j}, K to K_{α_j} and K^{-1} to $K_{\alpha_j}^{-1}$. Furthermore, this homomorphism will turn out to be isomorphism onto its image (in $U_q(\mathrm{sl}_{n+1})$).

As in the case $n = 1$, we can relate $U_q(\mathrm{sl}_{n+1})$ with the q-oscillator algebra $\mathcal{A}_q^{\mathrm{ext}}$. We consider the tensor product $\mathcal{A}_q^{\mathrm{ext}\otimes n+1}$ of $n+1$ copies of $\mathcal{A}_q^{\mathrm{ext}}$ whose set of generators is $\{a_1 a_1^\dagger, q^{\pm N_1/2}, \ldots, a_{n+1} a_{n+1}^\dagger, q^{\pm N_{n+1}/2}\}$. As in the case of $n = 1$, a possible Jordan–Schwinger realization of $U_q(\mathrm{sl}_{n+1})$ is achieved by mapping

$$\begin{array}{ll}
\mathrm{JS}_q(E_{\alpha_j}) = a_j^\dagger a_{j+1}, & \mathrm{JS}_q(F_{\alpha_j}) = a_{j+1}^\dagger a_j, \\
\mathrm{JS}_q(K_{\alpha_j}) = q^{(N_j - N_{j+1})/2}, & j = 1, \ldots n.
\end{array} \tag{14}$$

3. The representation theory of $U_q(\mathrm{sl}_{n+1})$

When a physical realization of the quantum algebra is considered, its representation theory plays a crucial role. The representations of the quantum algebra $U_q(\mathrm{sl}_2)$ are classified into three categories according to the value of q:

i. q is generic, that is, q can take any value except $q = 0, \pm 1$ and a root of unity,
ii. q is a root of unity,
iii. $q = 0$ (this case is also known as the crystal base).

It is known that for q generic, all finite-dimensional representations of $U_q(\mathrm{sl}_2)$ are completely reducible and the irreducible ones are classified in terms of highest weights. In particular, they can be regarded as deformation of the representations of the classical $U(\mathrm{sl}_2)$. When q is a root of unity, the representations of $U_q(\mathrm{sl}_2)$ become strikingly different from the classical case. They are not completely reducible and some finite-dimensional representations are not the highest weight ones.

As to $U_q(\mathrm{sl}_{n+1})$, its simple finite-dimensional representations are very similar to those of sl_{n+1} as long as q is not a root of unity. For $n = 1$, we have clearly all information about the simple representations of $U_q(\mathrm{sl}_2)$ (or equivalently $U_q(\mathrm{sl}_2)$-modules): for all positive integer m, there exist exactly two simple representations of $U_q(\mathrm{sl}_2)$ of dimension $m+1$ which correspond to each simple modules over sl_2. In general, when $n \neq 1$, the quantum algebra $U_q(\mathrm{sl}_{n+1})$ has $2^{|\Pi|}$ simple representations corresponding to each simple module for sl_{n+1}. These $2^{|\Pi|}$ modules arise from the choice of Π signs.

There exist different ways to describe the representations of $U_q(\mathrm{sl}_{n+1})$, but for our interest in chains with $n + 1$ sites, we use an approach carrying to irreducible finite-dimensional representations of $U_q(\mathrm{sl}_{n+1})$ by composing the Jordan–Schwinger realization with the Fock representation of the algebra $\mathcal{A}_q^{\mathrm{ext}\otimes n+1}$ (see also [11, section 5.3.4]). First, assume q is not a root of unity. The Fock representation of the algebra $\mathcal{A}_q^{\mathrm{ext}\otimes n+1}$ acting on the Hilbert space $\mathcal{H}^{\otimes n+1}$ with an orthonormal basis $|m_1, \dots, _{n+1}\rangle$ is determined by the formulae (6).

By the composition $\varphi := T \circ \mathrm{JS}_q$, an infinite-dimensional representation of the quantum algebra $U_q(\mathrm{sl}_2)$ can be formulated by linear operators on the space $\mathcal{D}^{\otimes n+1}$

$$\varphi : \quad U_q(\mathrm{sl}_{n+1}) \xrightarrow{JS_q} \mathcal{A}_q^{\mathrm{ext}\otimes n+1} \xrightarrow{T} \mathcal{L}(\mathcal{D}^{\otimes n+1}).$$

Furthermore, the basis elements $|m_1, \dots, m_{n+1}\rangle$ of $\mathcal{H}^{\otimes n+1}$ are represented as follows:

$$|m_1, \dots, m_{n+1}\rangle = \frac{T(a_1^\dagger)^{m_1}}{[m_1]!} \frac{T(a_2^\dagger)^{m_2}}{[m_2]!} \cdot \dots \cdot \frac{T(a_{n+1}^\dagger)^{m_{n+1}}}{[m_{n+1}]!} |0, \dots, 0\rangle.$$

So, the generators E_{α_j} and F_{α_j} of $U_q(\mathrm{sl}_2)$ for $j = 1, \dots, n+1$ are mapped by φ in this manner:

$$\begin{aligned} \varphi(E_{\alpha_j})\, |m_1, \dots, m_{n+1}\rangle &= T(a_j^\dagger) T(a_{j+1}) \frac{T(a_2^\dagger)^{m_2}}{[m_2]!} \cdot \dots \cdot \frac{T(a_{n+1}^\dagger)^{m_{n+1}}}{[m_{n+1}]!} |0, \dots, 0\rangle \\ &= \sqrt{[m_j + 1][m_{j+1}]}\, |m_1, \dots, m_j + 1, m_{j+1} - 1, \dots, m_{n+1}\rangle, \\ \varphi(F_{\alpha_j})\, |m_1, \dots, m_{n+1}\rangle &= \sqrt{[m_j][m_{j+1} + 1]}\, |m_1, \dots, m_j - 1, m_{j+1} + 1, \dots, m_{n+1}\rangle. \end{aligned} \tag{15}$$

For any positive integer number m, the linear subspace S^m spanned by the basis elements $|m_1, \dots, m_{n+1}\rangle$ with $m_1 + m_2 + \dots + m_{n+1} = m$ is invariant under the representation φ. So, the invariant subspace S^m of $\mathcal{H}^{\otimes n+1}$ is generated by the vectors

$$x_{m_1, m_2, \dots, m_{n+1}} := |m_1, \dots, m_{n+1}\rangle.$$

If we consider the Bargmann–Fock realization of $\mathcal{A}_q$ (that is, a realization of the Fock representation on the Hilbert space of entire holomorphic functions), then S^m represents the $\mathbb{C}$-vector space of all homogenous polynomials of $n+1$ variables $X_1, X_2, \ldots, X_{n+1}$ and degree m.

The restriction of T on the invariant subspace S^m is equivalent to the irreducible finite-dimensional representations $\varphi_{n,m}$ of $U_q(\mathrm{sl}_{n+1})$, $\varphi_{n,m} : U_q(\mathrm{sl}_{n+1}) \to \mathrm{End}(S^m)$ according to that $\varphi = \oplus_{m\in\mathbb{N}-\{0\}}\varphi_{n,m}$. By the action of φ given in (15), the generators E_{α_j}, F_{α_j}, K_{α_j} (with $j = 1, \ldots, n$) of $U_q(\mathrm{sl}_{n+1})$ act by $\varphi_{n,m}$ as follows:

$$
\begin{aligned}
E_{\alpha_j}\, x_{m_1,\ldots,m_{n+1}} &= \begin{cases} \sqrt{[m_j+1][m_{j+1}]}\, x_{m_1,\ldots,m_j+1,m_{j+1}-1,\ldots,m_{n+1}}, & \text{if } m_{j+1} > 0; \\ 0, & \text{if } m_{j+1} = 0. \end{cases} \\
F_{\alpha_j}\, x_{m_1,\ldots,m_{n+1}} &= \begin{cases} \sqrt{[m_j][m_{j+1}+1]}\, x_{m_1,\ldots,m_j-1,m_{j+1}+1,\ldots,m_{n+1}}, & \text{if } m_j > 0; \\ 0, & \text{if } m_j = 0. \end{cases} \\
K_{\alpha_i}\, x_{m_1,m_2,\ldots,m_{n+1}} &= q^{m_1-m_{i+1}}\, x_{m_1,m_2,\ldots,m_{n+1}}.
\end{aligned}
\tag{16}
$$

Every $x_{m_1,m_2,\ldots,m_{n+1}}$ is a weight vector and spans every nonzero weight space in S^m (which therefore has dimension 1). In particular, all E_{α_i} annihilate $\bar{x}_{m,0,0,\ldots,0}$. Up to the scalar multiplication this is the only vector with this property. Hence, S^m is an irreducible representation of $U_q(\mathrm{sl}_{n+1})$ (for every $n \geqslant 1$).

Actually, the construction of the representation space S^m holds even if q is a root of unity, but in general the irreducibility of S^m is lost. For instance, for $n = 1$, if the order d of q is bigger than $m+1$, then S^m is simple and the map $\varphi_{1,m}$ acts in the same way described above; if d is smaller than $m+1$, then no simple finite-dimensional representation exists; if $d = m+1$ we should discuss other conditions.

4. Deformed chains and perfect state transfer

We are now able to approach the study of state transfer in a chain of q-deformed oscillators. We consider the following protocol. The ends of the quantum chain, i.e. the first and the $(n+1)$th site, are assigned respectively to the sender and the receiver. The remaining $n-1$ oscillators constitute the communication channel. The quantum chain is initialized in the vacuum state $|0\rangle|0\rangle^{\otimes n-1}|0\rangle$, defined by $T(a_j)|0\rangle = 0$. The transfer protocol begins when the sender prepares her oscillator in a qu$D$it state $|\psi\rangle = \sum_{m=0}^{D-1} c_m|m\rangle$ where, according to the Fock representation (6), $|m\rangle = K_m{}^{-1/2}T(a_1^\dagger)^m|0\rangle$, with

$$
K_m = [m][m-1]\cdots[2][1]. \tag{17}
$$

Then the quantum chain evolves according to the chain Hamiltonian (1). Note that the Hamiltonian (1) preserves the total number of excitations in the q-deformed chain. We refer to the manifold of states of the chain with m excitations as the mth Fock layer. It follows that the chain dynamics does not mix Fock layer of different degree. After a transfer time t the sender instantaneously decouples the $(n+1)$th oscillator from the rest of the chain. At this point, the receiver can apply a suitable phase gate $\mathcal{U} = \sum_{m=0}^{D-1} \mathrm{e}^{\mathrm{i}\phi_m}|m\rangle\langle m|$ on her oscillator to maximize the transfer fidelity [1, 4]. This local transformation at the receiver site is independent of the state encoded by the sender and is only determined by the chain Hamiltonian, its length and the transfer time t. The reduced state of the oscillator at the receiver site is hence denoted by $\rho(t)$. To evaluate the quality of the state transfer, we consider the transfer fidelity $F(t) = \langle\psi|\rho(t)|\psi\rangle$, averaged over all possible input states.

In the classical case of a chain of spin-1/2, necessary and sufficient conditions for obtaining a perfect state transfer have been determined; see e.g. [12] for a review. In particular, it is possible to reach a perfect transfer if the coupling constants in the Hamiltonian (1) are modulated according to

$$
J_j = \lambda\sqrt{j(n+1-j)}. \tag{18}
$$

In this way, the chain evolution is formally equivalent to a rotation about the x-axis of a 'big spin' expressing a collective degree of freedom of the quantum chain [2]. The same choice of the coupling constants allows perfect state transfer in a bosonic chain [4]. In this case, in each Fock layer the chain evolution is equivalent to a rotation of a collective spin about the x-axis. The perfect state transfer can be seen as a consequence of the algebraic identity

$$\mathrm{e}^{\mathrm{i}\pi S_x} S_- \mathrm{e}^{-\mathrm{i}\pi S_x} = S_+, \tag{19}$$

where S_+, S_-, S_z are the collective spin operators in the mth Fock layer, and the transfer time is independent of the length of the chain and of the order of the Fock layer and equals $t = \pi/\lambda$.

Using the theory of representations of $U_q(\mathrm{sl}_{n+1})$ one can explicitly show that the choice of the coupling constants (18) allows perfect state transfer in a chain of q-deformed oscillators if quantum information is encoded using only the vacuum state and the first Fock layer. However, if higher Fock layer is included in the encoding, the choice (18) is no longer sufficient to allow perfect state transfer in a chain of q-deformed oscillators. Indeed, the effects of nonlinearity introduced by the q-deformation manifest themselves if two or more excitations are present in the quantum chain.

4.1. PST in the first Fock layer

Here we consider the case of the transfer of a qubit state encoded as $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$. In this case, the chain dynamics only involves the vacuum state and the first Fock layer.

Let us start to discuss the case when n, m are both equal to 1, that is, we have a network with two sites (so the quantum algebra $U_q(\mathrm{sl}_2)$ as the mathematical model) and just one excitation. Thus, by considering the representation map $\varphi_{1,1} : U_q(\mathrm{sl}_2) \to S^1$, the matrices determined by the action (by $\varphi_{1,1}$) of generators of $U_q(\mathrm{sl}_2)$

$$\varphi_{1,1}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \varphi_{1,1}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \varphi_{1,1}(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

coincide with the generators of sl_2. As in the classical case (see [2]), let us choose three variables S_x and S_y in $U_q(\mathrm{sl}_2)$ as follows:

$$S_x := \frac{E+F}{2}, \qquad S_y := \frac{E+F}{2\mathrm{i}},$$

and $S_+, S_- \in U_q(\mathrm{sl}_2)$ as

$$S_+ := S_x + \mathrm{i}S_y, \qquad S_- := S_x - \mathrm{i}S_y.$$

By applying the representation map to these new variables, we can easily note that $\varphi_{1,1}(S_x)$, $\varphi_{1,1}(S_y)$, $\varphi_{1,1}(S_z)$ coincide with the generators of the Lie algebra su(2) of traceless skew-Hermitian matrices and $\varphi_{1,1}(S_+)$, $\varphi_{1,1}(S_-)$ with the Pauli matrices, that is, with the generators of the (special unitary) Lie group $SU(2)$ of unitary matrices with unit determinant.

We now consider the case of a chain of $n+1$ q-deformed oscillators. The related mathematical setting is formed by the quantum algebra $U_q(\mathrm{sl}_{n+1})$ with the generators E_{α_j}, F_{α_j}, K_{α_j} (for $j = 1, \ldots, n$) and by the representation S^m of all homogenous polynomials of $n+1$ variables and degree m. A possible strategy is that of generalizing the previous result shown for $n, m = 1$ to this framework. First, we can show the analogous relations (19) for the case of $n+1$ sites and 1 excitation (with S^1 being the related representation).

Proposition 4.1. *Let $\varphi_{n,1}$ denote the representation map $\varphi_{n,1} : U_q(sl_2) \to End(S^1)$ taking the generators of $U_q(sl_{n+1})$, E_{α_j}, F_{α_j}, K_{α_j}, respectively to the $(n+1) \times (n+1)$ matrices $\varphi_{n,1}(E_{\alpha_j})$, $\varphi_{n,1}(F_{\alpha_j})$, $\varphi_{n,1}(K_{\alpha_j}) \in M_{n+1}(\mathbb{C})$.*

Let us set $S_x, S_y \in U_q(sl_{n+1})$ *as*

$$S_x := \sum_{j=1}^{n} \sqrt{j(n-j+1)}\, \frac{E_{\alpha_j} + F_{\alpha_j}}{2}, \qquad S_y := \sum_{j=1}^{n} \sqrt{j(n-j+1)}\, \frac{E_{\alpha_j} - F_{\alpha_j}}{2i}, \tag{20}$$

and $S_+, S_- \in U_q(sl_{n+1})$ *as*

$$S_+ := S_x + \mathrm{i}S_y, \qquad S_- := S_x - \mathrm{i}S_y. \tag{21}$$

Then, the relation

$$\exp(\mathrm{i}t\varphi_{n,1}(S_x))\, \varphi_{n,1}(S_-) \exp(-\mathrm{i}t\varphi_{n,1}(S_x)) = \varphi_{n,1}(S_+) \tag{22}$$

holds for the time value $t = \pi$.

Proof. According to the relations (16) applied to the $n+1$ basis vectors of S^1, $x_{1,0,\ldots,0}, \ldots, x_{0,0,\ldots,1}$, the matrices $\varphi_{n,1}(E_{\alpha_i}),\ \varphi_{n,1}(F_{\alpha_i}),\ \varphi_{n,1}(K_{\alpha_i})$ are

$$\varphi_{n,1}(E_{\alpha_1}) = \begin{pmatrix} 0 & 1 & 0 & \ldots 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & & \ldots & 0 \\ 0 & & \ldots & 0 \end{pmatrix}, \ldots, \varphi_{n,1}(E_{\alpha_n}) = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \ldots & 1 \\ 0 & & \ldots & 0 \end{pmatrix}$$

$$\varphi_{n,1}(F_{\alpha_1}) = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ 0 & & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & & \ldots & 0 \end{pmatrix}, \ldots, \varphi_{n,1}(F_{\alpha_n}) = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0\ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix},$$

$$\varphi_{n,1}(K_{\alpha_1}) = \begin{pmatrix} q & & \ldots & 0 \\ 0 & q^{-1} & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \ldots & 1 \end{pmatrix}, \ldots, \varphi_{n,1}(K_{\alpha_n}) = \begin{pmatrix} 1 & & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots q & 0 \\ 0 & 0 & \ldots & q^{-1} \end{pmatrix}.$$

By choosing S_x, S_y as in (20) and S_+, S_- as in (21), the corresponding matrices

$$\varphi_{n,1}(S_+) = \begin{pmatrix} 0 & \sqrt{n} & 0 & \ldots & 0 \\ 0 & 0 & \sqrt{2(n-1)} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \sqrt{n} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{23}$$

$$\varphi_{n,1}(S_-) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \sqrt{n} & 0 & 0 & \ldots & 0 \\ 0 & \sqrt{2(n-1)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{n} & 0 \end{pmatrix} \tag{24}$$

are shown to be compatible with the classical case, so the statement is easily proved. □

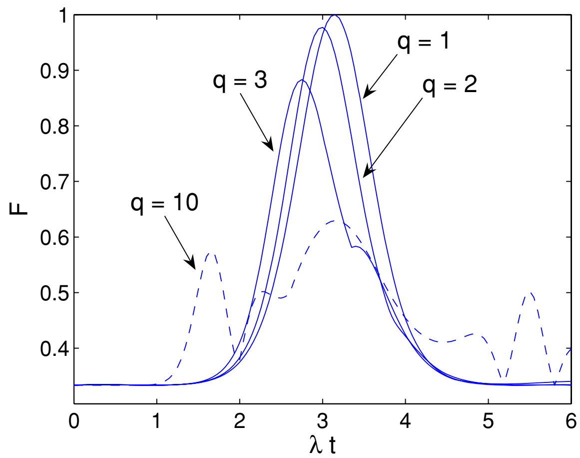

Figure 1. The average fidelity of the state transfer versus the strength of the interaction λt in the second Fock layer, for a chain of 10 q-deformed bosons. Different lines refer to different values of the deformation parameter. Note that the dynamics is symmetric under the exchange $q \leftrightarrow q^{-1}$.

4.2. State transfer in higher Fock layer

Here we consider the case of a qu*D*it encoding exploiting states with higher number of excitations. We study the transfer of one qutrit encoded at the sender site in a state of the form $|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$ and numerically evaluate the average transmission fidelity as a function of the transfer time and the deformation parameter, when the coupling constants are chosen according to (18). For $q = 1$ the 'classical' bosonic chain is recovered, and the choice of coupling constants is optimal. Deviations from this classical behavior appear as long as $q \neq 1$. The q-deformation in the algebraic structures induces a nonlinear perturbation in the spectrum of the bosonic chain. The nonlinear effects manifest themselves when two or more excitations are present in the quantum chain. This will in general affect the fidelity of the state transfer with respect to the undeformed bosonic chain.

Figures 1 shows the average transfer fidelity as a function of the (adimensional) transfer time λt, for a chain of 10 q-deformed oscillators. The undeformed chain, recovered for $q = 1$, allows perfect state transfer after a minimal transfer time $\lambda t = \pi$. For increasing value of the nonlinearity parameter q, the maximum average fidelity decreases, while the (non-perfect) state transfer is generally faster. Figure 2 shows the maximum average fidelity of the state transfer and the corresponding optimal transfer time as a function of the deformation parameter. The analysis is restricted to a temporal window $\lambda t \in [0, 2\pi]$, corresponding to the period of the undeformed dynamics [4]. Note that, from the form of the q-number (5), the dynamics is symmetric under the exchange $q \leftrightarrow q^{-1}$.

In some cases, the introduction of the q-deformation at the algebraic level can be used to interpolate, varying the value of the deformation parameter q, between the 'classical' cases of a chain of spin-1/2 and a bosonic chain. For instance, by choosing $q = \mathrm{e}^{\pm \mathrm{i}\pi/d}$, for any integer d, it is possible to show that the Fock space is the direct sum of d-dimensional subspace, which are not connected by the ladder operators [13]. This is a consequence of the deformed commutation relations, which implies $T(a_k)^d = 0$, $T(a_k{}^{\dagger})^d = 0$. From this point of view, one can consider the chain of deformed oscillators with $q = \exp(\pm \mathrm{i}\pi/d)$ as a chain of d-level systems with non-equally spaced energy levels. Hence, by varying the integer d, one can

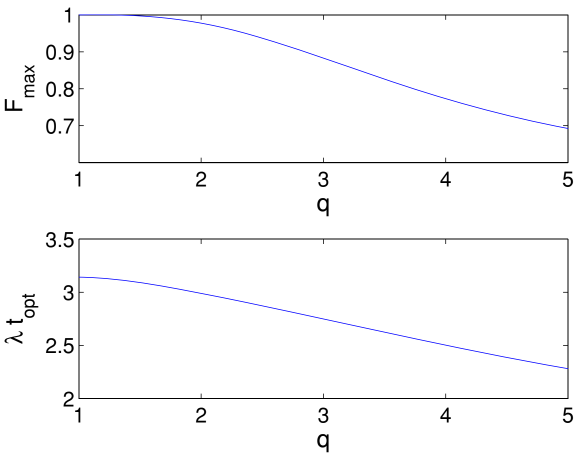

Figure 2. For a chain of 10 q-deformed bosons, the figure shows the maximum average fidelity (top) of the state transfer in the second Fock layer and the corresponding optimal (adimensional) transfer time λt_{opt} (bottom), as a function of the deformation parameter q. The analysis is restricted to a temporal window $\lambda t \in [0, 2\pi]$, corresponding to the period of the undeformed dynamics [4].

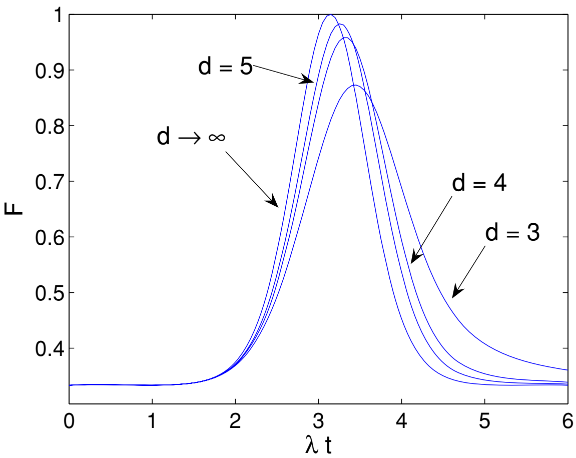

Figure 3. The average fidelity of the state transfer versus the strength of the interaction λt in the second Fock layer, for a chain of 10 q-deformed bosons. The deformation parameter is $q = \mathrm{e}^{\mathrm{i}\pi/d}$. Different lines refer to different values of the deformation parameter. Note that the classical bosonic case is recovered in the limit $d \to \infty$.

interpolate between the spin-$\frac{1}{2}$ case, obtained for $d = 2$, and the bosonic case, recovered in the limit of $d \to \infty$. We consider the case $d > 2$, since for $d = 2$ the condition $T(a^{\dagger})^2 = 0$ (Pauli principle) avoids two excitations on the same site. Figure 3 shows the average fidelity of the state transfer for a chain of q-deformed oscillators as a function of the transfer time, for several values of the effective Hilbert space dimension d. The minimal dimension in which the two-excitation encoding can be defined is $d = 3$. Note that the bosonic limit is recovered for $d \to \infty$, in which case perfect state transfer happens for a minimal transfer time $\lambda t = \pi$. Finite values of d lead to a smaller transfer fidelity and a longer optimal time transfer. Figure 4

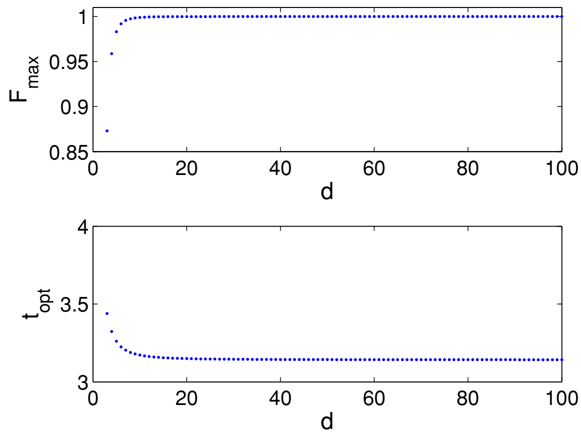

Figure 4. For a chain of 10 q-deformed bosonic oscillators with $q = e^{i\pi/d}$, the figure shows the maximum average fidelity (top) in the second Fock layer, and the corresponding optimal (adimensional) transfer time λt_{opt} (bottom), as a function of the deformation parameter d. Note that the classical bosonic case is recovered in the limit $d \to \infty$.

shows the maximum average transfer fidelity and the corresponding optimal transfer time as a function of the effective Hilbert space dimension d.

5. Conclusions

We have considered the issue of state transfer through a quantum chain of q-deformed oscillators. For real values of the deformation parameter, the physical consequence of the algebraic deformation is the appearance of non-harmonicity in the energy spectrum of the chain. The q-deformation can be hence interpreted as a formal way to describe a bosonic chain with nonlinear interactions. If only states with one excitation are involved, the nonlinearities do not play any role and the q-deformed dynamics is identical to its classical, linear, counterpart. More generally, if the considered protocol involves states of the chain with two or more excitations, we have found that the nonlinear effects decrease the fidelity of the state transfer, while however shortening the optimal transfer time. Similar results were recently presented in [4], where the state transfer through a bosonic chain described by the (nonlinear) Bose–Hubbard Hamiltonian was considered. In our analysis, we have chosen the coupling constants according to (18), a choice which is optimal in the undeformed case. Clearly, alternative q-dependent choices of the coupling constants could lead to better performances.

Finally, if the deformation parameter is chosen to be a root of the unity of order d, the q-deformed oscillator can be used to simulate a d-level quantum system with non-equally spaced stationary level. In this case, varying the deformation parameter from $d = 2$ to $d \to \infty$ one can describe a family of quantum chain interpolating between a chain of spin-1/2 and the bosonic chain.

Acknowledgments

The work of CL and SM is partially supported by EU through the FET-Open Project HIP (FP7-ICT-221899).

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